

WEIL REPRESENTATIONS OVER FINITE FIELDS AND SHINTANI LIFT

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ABSTRACT. Let $\mathrm{Sp}_V(F)$ be the group of isometries of a symplectic vector space V over a finite field F of odd cardinality. The group $\mathrm{Sp}_V(F)$ possesses distinguished representations—the Weil representations. We know that they are compatible with base change in the sense of Shintani for a finite extension F'/F . The result is also true for the group of similitudes of V .

1. INTRODUCTION

Let F be a finite field of odd cardinality q , and let ψ be a non-trivial character of F . Consider a symplectic vector space V over F of finite dimension and write \mathbf{Sp}_V for its group of isometries seen as an algebraic group over F . To ψ is attached a canonical class of representations of $\mathbf{Sp}_V(F)$, the Weil representations $W_\psi[\mathrm{Ge}]$. Let F' be a finite extension of F with Frobenius automorphism σ . In this paper, we establish the behavior of W_ψ with respect to Shintani lifting from F to F' . We recall that there is a norm map N yielding a bijection from the set of $\mathbf{Sp}_V(F')$ -conjugacy classes of $\sigma \ltimes \mathbf{Sp}_V(F')$, a subset of $\mathrm{Gal}(F'/F) \ltimes \mathbf{Sp}_V(F')$, onto the set of conjugacy classes of $\mathbf{Sp}_V(F)$. Now set $\psi' = \psi \circ \mathrm{tr}_{F'/F}$.

Theorem. *There is a canonical extension $\widetilde{W}_{\psi'}$ of $W_{\psi'}$ to $\mathrm{Gal}(F'/F) \ltimes \mathbf{Sp}_V(F')$ such that*

$$(\star) \quad \mathrm{tr} \widetilde{W}_{\psi'}(\sigma, g) = \mathrm{tr} W_\psi(Ng)$$

for any $g \in \mathbf{Sp}_V(F')$.

We actually give an explicit model for $\widetilde{W}_{\psi'}$, using the Schrödinger model of $W_{\psi'}$ (cf. §4). Note that our results are in fact more general, in that we consider norm maps for any power of σ : the corresponding statement is in §4. We also establish the analogous results in §4 for the Weil representation of the group \mathbf{GSp}_V of similitudes of (V, \langle, \rangle) —the class of that representation does not depend on the choice of ψ . With the same methods, we can prove that the Weil representations of general linear groups and unitary groups defined by Gérardin in [[Ge], §2 and §3] are compatible with Shintani lifting as well. We shall come back to those cases, with applications, in future work.

In fact the character relation (\star) in the theorem is valid for a pair (σ, g) , where g is in the semi-direct product $\mathbf{Sp}_V(F') \ltimes \mathbf{H}_V(F')$. But the identity is $0 = 0$ unless (σ, g) is conjugate to (σ, g') with g' in $\mathbf{Sp}_V(F') \times \mathbf{Z}_V(F')$, \mathbf{Z}_V being the centre of \mathbf{H}_V (see §5). So in effect, we are reduced to proving (\star) for a fixed g in $\mathbf{Sp}_V(F')$, or more conveniently for a fixed norm h in $\mathbf{Sp}_V(F)$.

We proceed by induction on $2n = \dim V$, allowing the field F to vary. If h belongs to some proper parabolic subgroup of $\mathbf{Sp}_V(F)$, we use the mixed Schrödinger model (§4) to reduce to a smaller dimension (§8). If h stabilizes a decomposition $V = V_1 \oplus V_2$ of V into a direct sum of two non-zero symplectic subspaces, again we are reduced to a smaller dimension (§6, §8). The remaining case is when h is a regular element of a maximally elliptic torus $\mathbf{T}(F)$ of $\mathbf{Sp}_V(F)$. More concretely, V is a one-dimensional skew-hermitian vector space over a finite extension E of F of degree n and h is an element in E^\times (acting on V), of norm 1 in the subfield E_+ such that $[E : E_+] = 2$. That case is treated in §9 and §10 with some explicit computations.

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2. NOTATION

Throughout the paper, F is a finite field of **odd** cardinality q , and F' is a finite extension of F of degree m ; we put $\Gamma = \mathrm{Gal}(F'/F)$. We also fix an algebraic closure \overline{F} of F' , and write σ for the Frobenius automorphism $x \mapsto x^q$ of \overline{F} ; it restricts to the Frobenius automorphism of F' over F , for which we also write σ . For any positive integer d , we let F_d be the degree d extension of F in \overline{F} ; thus $F' = F_m$.

If σ acts on a set X , we write X_σ for the set of fixed points of σ in X , we use a similar notation for powers of σ . If G is a group, we write $C(G)$ for the vector space of complex valued class functions on G ; if $G \ltimes H$ is a semi-direct product group, then the law will be given by $(g, h) \cdot (g', h') = (gg', h \cdot g(h'))$, where $g(h')$ denotes the action of $g \in G$ on the element h' of the invariant subgroup H .

3. NORM MAPS

Let i be an integer, $0 \leq i \leq m-1$, write d for the greatest common divisor of m and i ; put $i = dj$, $m = d\mu$ for some integers j and μ . We choose an integer t such that $ti \equiv d \pmod{m}$.

Let \mathbf{G} be a connected linear algebraic group over the field F . We consider the semi-direct product $\mathrm{Gal}(\overline{F}/F) \ltimes \mathbf{G}(\overline{F})$. In [Gy], Gyoja constructs a norm map $N_{i,t}$ from $\sigma^i \ltimes \mathbf{G}(F')$ to $\mathbf{G}(F_d)$ in the following way:

For g in $\mathbf{G}(F')$, choose $\alpha = \alpha(g)$ in $\mathbf{G}(\overline{F})$ such that

$$(1, \alpha^{-1} \sigma^d(\alpha)) = (\sigma^{-it}, 1) \cdot (\sigma^i, g)^t$$

and let

$$N_{i,t}(\sigma^i, g) = \alpha(g \sigma^i(g) \cdots \sigma^{i(\mu-1)}(g)) \alpha^{-1}.$$

That element $N_{i,t}(\sigma^i, g)$ does belong to $\mathbf{G}(F_d)$, and its conjugacy class in $\mathbf{G}(F_d)$ does not depend on the choice of α . Moreover, Gyoja shows that $N_{i,t}$ induces a bijection from the set of $\mathbf{G}(F')$ -conjugacy classes in $\sigma^i \ltimes \mathbf{G}(F')$ onto the set of conjugacy classes in $\mathbf{G}(F_d)$. It is immediate that this bijection is σ -equivariant.

Remarks:

(i) For $i = t = 1$, we recover the classical Shintani norm map[[D1], [S]]. Note that $N_{i,t}$ does depend on the choice of t ; for instance, it can be proved that $N_{1,m+1} = Sh_{F/F} \circ N_{1,1}$, where $Sh_{F/F}$ is the notation for the Shintani self-lift of [D1].

(ii) Putting $\tau = \sigma^d$, which is the Frobenius automorphism for \overline{F}/F_d , we see that for $\sigma^i(g) = \tau^j(g)$, the norm $N_{i,t}(\sigma^i, g)$ is the same as $N_{j,t}(\tau^j, g)$, thus we can always reduce our considerations to the case where i is prime to m , at the cost of allowing a change of base field from F to F_d .

(iii) Assume that \mathbf{G} is commutative; then for g in $\mathbf{G}(F')$, we have $N_{i,t}(\sigma^i, g) = g \sigma^i(g) \cdots \sigma^{i(\mu-1)}(g)$, in other words, this is simply the usual norm of g from $\mathbf{G}(F')$ to $\mathbf{G}(F_d)$.

Composing with $N_{i,t}$ gives a vector space isomorphism $N_{i,t}$ of $C(\mathbf{G}(F_d))$ onto the vector space $C(\sigma^i \ltimes \mathbf{G}(F'))$ of complex valued functions on $\sigma^i \ltimes \mathbf{G}(F')$ which are invariant under conjugation by $\mathbf{G}(F')$. It induces an isomorphism

of $C(\mathbf{G}(F_d))_\sigma$ onto the vector space $C(\sigma^i \ltimes \mathbf{G}(F'))_\sigma$ of complex valued functions on $\sigma^i \ltimes \mathbf{G}(F')$ which are invariant under conjugation by $\Gamma \ltimes \mathbf{G}(F')$.

When \mathbf{G} is abelian, and χ is a character of $\mathbf{G}(F)$, we get a character $\tilde{\chi}'$ of $\Gamma \ltimes \mathbf{G}(F')$ by composing χ with the usual norm N from $\mathbf{G}(F')$ to $\mathbf{G}(F)$ and extending trivially on Γ . So for $g \in \mathbf{G}(F')$, we have

$$\tilde{\chi}'(\sigma, g) = \chi(N(g)).$$

When \mathbf{G} is non-abelian, and χ is the character of an irreducible representation of $\mathbf{G}(F)$, it is *not* generally the case that $N_{i,t}(\chi)$ is the restriction to $\sigma^i \ltimes \mathbf{G}(F')$ of some character of a representation of $\Gamma \ltimes \mathbf{G}(F')$. However this paper is concerned with a situation where it is indeed the case.

For later use, we shall recall some results of Gyoja in [Gy]:

Lemma 3.1. (i) For any $\chi, \chi' \in C(\mathbf{G}(F_d))$, we have $\langle \chi, \chi' \rangle = \langle N_{i,t}(\chi), N_{i,t}(\chi') \rangle$, where $\langle \chi, \chi' \rangle := \frac{1}{|G(F_d)|} \sum_{x \in G(F_d)} \chi(x) \overline{\chi'(x)}$ and $\langle N_{i,t}(\chi), N_{i,t}(\chi') \rangle := \frac{1}{|\sigma^i \ltimes G(F')|} \sum_{y \in G(F')} \chi(N_{i,t}(\sigma^i, y)) \overline{\chi'(N_{i,t}(\sigma^i, y))}$.

(ii) Through the lifting maps $N_{i,t}$ by allowing i to vary from 0 to $m-1$, we can decompose $C(\text{Gal}(F'/F)\mathbf{G}(F'))$ as the direct sum $\bigoplus_{i=0}^{m-1} C(\mathbf{G}(F')_{\sigma^i})_\sigma$.

(iii) The above decomposition is compatible with the usual induction map, the restriction map, the product map, etc. For example, if \mathbf{H} is a connected algebraic subgroup of \mathbf{G} defined over F , then for $\chi \in C(\text{Gal}(F'/F)\mathbf{G}(F'))$ such that $\chi|_{\sigma^i \ltimes \mathbf{G}(F')} = N_{i,t}(\chi')$ for some $\chi' \in C(\mathbf{G}(F_d))_\sigma$, we have $\text{Res}_{\text{Gal}(F'/F)\mathbf{H}(F')}^{\text{Gal}(F'/F)\mathbf{G}(F')}(\chi) = N_{i,t} \circ \text{Res}_{\mathbf{H}(F_d)}^{\mathbf{G}(F_d)}(\chi')$.

4. EXTENDED WEIL REPRESENTATION

As in the introduction, we fix a symplectic vector space V over F , and write $2n$ for its dimension, \langle, \rangle for the symplectic form on V . We see V as a linear algebraic group, denoted by the bold letter \mathbf{V} , and similarly for the group \mathbf{Sp}_V of isometries of \mathbf{V} , the group \mathbf{GSp}_V of similitudes of V .

Let \mathbf{H}_V be the Heisenberg group over F associated to V : for each F -algebra R , $\mathbf{H}_V(R)$ is the set $\mathbf{V}(R) \oplus R$, endowed with the group law

$$(v_1, t_1)(v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \frac{1}{2}\langle v_1, v_2 \rangle_R),$$

where the form \langle, \rangle_R is obtained by scalar extension. Then \mathbf{H}_V is a non-abelian connected algebraic group over F , with centre \mathbf{Z}_V such that $\mathbf{Z}_V(R) = \{(0, x) | x \in R\}$.

Fix a non-trivial character ψ of F , and put $\psi' = \psi \circ \text{tr}_{F'/F}$. To ψ is associated the Weil representation of $\mathbf{Sp}_V(F)\mathbf{H}_V(F)$ —it is in fact an isomorphism class of representations. We write ρ for that Weil representation, and $\text{tr}(\rho)$ for its character. Similarly to ψ' is associated the Weil representation ρ' of $\mathbf{Sp}_V(F')\mathbf{H}_V(F')$. Indeed for each positive integer d , we have a Weil representation ρ_d of $\mathbf{Sp}_V(F_d)\mathbf{H}_V(F_d)$ associated to the character $\psi_d = \psi \circ \text{tr}_{F_d/F}$. Our main result is the following:

Theorem 4.1. *There is a unique extension $\tilde{\rho}'$ of ρ' to $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{H}_V(F')$ such that, for integers i, t, d as in §3, and $g \in \mathbf{Sp}_V(F')\mathbf{H}_V(F')$, we have*

$$(\star) \quad \text{tr} \tilde{\rho}'(\sigma^i, g) = \text{tr} \rho_d(N_{i,t}(\sigma^i, g)).$$

In particular, for $i = t = 1$, we obtain

$$\text{tr} \tilde{\rho}'(\sigma, g) = \text{tr} \rho(N_{1,1}(\sigma, g)),$$

i.e. the Weil representation is “compatible” with Shintani lifting.

As indicated in the introduction, this will be proved progressively. In this §4, we use the Schrödinger model of ρ' to construct an extension $\tilde{\rho}'$ such that $\text{tr} \tilde{\rho}'(\sigma) = \text{tr} \rho(1)$; note that $N_{1,t}(\sigma) = 1$ for all possible t 's. Then by Clifford theory, $\tilde{\rho}'$ is the unique extension satisfying this simple character relation, so the remaining problem will be to prove (\star) in general. For this purpose, in the following section §5, we examine the support of the character $\tilde{\rho}'$; in §6 and §7, we consider the restriction of $\tilde{\rho}'$ to some interesting subgroups; the proof of (\star) will be reduced to a very special case, and we treat this special case in §9 and §10.

Firstly admitting the theorem, let us derive a consequence for the groups of symplectic similitudes. Put $\pi = \text{Ind}_{\mathbf{Sp}_V(F)}^{\mathbf{GSp}_V(F)} \rho|_{\mathbf{Sp}_V(F)}$; it is the Weil representation of $\mathbf{GSp}_V(F)$ which is independent (up to isomorphism) of the choice of ψ [Ge]; similarly as in §4, for each factor d of m , we denote the corresponding Weil representation of $\mathbf{GSp}_V(F_d)$ by π_d , and also write π' for $d = m$.

Theorem 4.2. *There is a unique extension $\widetilde{\pi}'$ of π' to $\Gamma \ltimes \mathbf{GSp}_V(F')$ such that, for integers i, t, d as in §3, and $g' \in \mathbf{GSp}_V(F')$, we have*

$$(\star') \quad \mathrm{tr} \widetilde{\pi}'(\sigma^i, g') = \mathrm{tr} \pi_d(\mathrm{N}_{i,t}(\sigma^i, g')).$$

Moreover, the induced representation of $\Gamma \ltimes \mathbf{GSp}_V(F')$ from the representation $\widetilde{\rho}'$ of $\Gamma \ltimes \mathbf{Sp}_V(F')$ satisfies the desired conditions.

Proof. Uniqueness comes from Lemma 3.1 (ii), and by (iii) in the same lemma and Theorem 4.1, we see $[\mathrm{Ind}_{\mathrm{Gal}(F'/F)\mathbf{Sp}_V(F')}^{\mathrm{Gal}(F'/F)\mathbf{GSp}_V(F')}(\mathrm{tr} \widetilde{\rho}')](\sigma^i, g') = [\mathrm{N}_{i,t}(\mathrm{Ind}_{\mathbf{Sp}(F_d)}^{\mathbf{GSp}(F_d)} \mathrm{tr} \rho_d)](\sigma^i, g')$; in this equality, the second term is equal to $\mathrm{tr} \pi_d(\mathrm{N}_{i,t}(\sigma^i, g'))$, so the results follow. \square

Now let us fix a complete polarisation $V = X \oplus X^*$ of V , so that X, X^* are two Lagrangian subspaces of V . We denote the corresponding algebraic groups over F by \mathbf{X}, \mathbf{X}^* and \mathbf{V} respectively and write ϵ' for the unique non trivial quadratic character of F'^\times . Then the Weil representation ρ' of $\mathbf{Sp}_V(F')\mathbf{H}_V(F')$ can be realized in the space $\mathbb{C}[\mathbf{X}^*(F')]$ of complex functions on $\mathbf{X}^*(F')$ by the following formulas [cf. [Ge]]:

$$(1) \quad \rho'(1, (x + x^* + k))f(y^*) = \psi'(k + \langle y^*, x \rangle)f(x^* + y^*),$$

$$(2) \quad \rho'\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right)f(y^*) = \psi'\left(\frac{\langle by^*, y^* \rangle}{2}\right)f(y^*),$$

$$(3) \quad \rho'\left(\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, 1\right)f(y^*) = \epsilon'(\det(a))f(a^*y^*),$$

$$(4) \quad \rho'\left(\begin{pmatrix} 0 & c' \\ c & 0 \end{pmatrix}, 1\right)f(y^*) = \gamma(\psi')^{-n} \epsilon'(\det(c)) \int_{\mathbf{X}^*(F')} f(x^*) \psi'(\langle x^*, c^{-1}y^* \rangle) dx^*,$$

where $b \in \mathrm{Hom}(\mathbf{X}^*(F'), \mathbf{X}(F'))$, $a \in \mathrm{Aut}(\mathbf{X}(F'))$, and $a^* \in \mathrm{Aut}(\mathbf{X}^*(F'))$ is the adjoint of a with respect to the bilinear form $\mathbf{X}(F') \times \mathbf{X}^*(F') \rightarrow F'$ given by $(x, x^*) \mapsto \langle x, x^* \rangle$, and finally $c \in \mathrm{Isom}(\mathbf{X}^*(F'), \mathbf{X}(F'))$, $c' \in \mathrm{Isom}(\mathbf{X}(F'), \mathbf{X}^*(F'))$.

Let I_σ be the automorphism of $\mathbb{C}[\mathbf{X}^*(F')]$ given by

$$I_\sigma(f)(x) = f(\sigma^{-1}(x)) \text{ for } f \in \mathbb{C}[\mathbf{X}^*(F')], x \in \mathbf{X}^*(F').$$

It is easily verified on the formulas (1) to (4) that

$$I_\sigma \rho'(g) I_\sigma^{-1} = \rho'(\sigma(g)) \text{ for } g \in \mathbf{Sp}_V(F')\mathbf{H}_V(F');$$

in formulas (1) and (2), one uses the facts that ψ' is σ -invariant and that the symplectic form on $\mathbf{V}(F')$ is σ -equivariant; in formulas (3) and (4), one uses moreover that ϵ' is also σ -invariant.

Since (I_σ^m) is the identity, it follows that there is a unique extension of the action ρ' of $\mathbf{Sp}_V(F')\mathbf{H}_V(F')$ on $\mathbb{C}[\mathbf{X}^*(F')]$ to an action $\widetilde{\rho}'$ of $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{H}_V(F')$ such that σ acts via I_σ . By the formulas for I_σ , $\mathrm{tr} \widetilde{\rho}'(\sigma) = q^n$ which is also $\mathrm{tr} \rho(1)$.

5. SUPPORT OF THE CHARACTER OF THE EXTENDED WEIL REPRESENTATION

It is a result of [[Ge], p.84-85] that $\check{\rho} \otimes \rho$ is isomorphic to the representation of $\mathbf{Sp}_V(F)\mathbf{H}_V(F)$ induced from the trivial representation of $\mathbf{Sp}_V(F)\mathbf{Z}_V(F)$; in particular, the character of ρ is 0 outside the conjugates of $\mathbf{Sp}_V(F)\mathbf{Z}_V(F)$. We establish the analogous fact for $\widetilde{\rho}'$.

Proposition 5.1. *$\check{\rho}' \otimes \widetilde{\rho}'$ is isomorphic to the representation of $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{H}_V(F')$ induced from the trivial character of $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{Z}_V(F')$.*

Proof. The representation $\widetilde{\lambda}' = \mathrm{Ind}_{\mathrm{Gal}(F'/F)\mathbf{Sp}_V(F')\mathbf{H}_V(F')}^{\mathrm{Gal}(F'/F)\mathbf{Sp}_V(F')\mathbf{H}_V(F')} 1$ can be realized in $\mathbb{C}[\mathbf{V}(F')]$ by the following formulas:

$$(5) \quad \widetilde{\lambda}'(h)(F)(v) = F(v + v_0) \text{ for } h \in \mathbf{H}_V(F') \text{ with projection } v_0 \text{ on } \mathbf{V}(F'),$$

$$(6) \quad \widetilde{\lambda}'(s)(F)(v) = F(s^{-1}v) \text{ for } s \in \mathbf{Sp}_V(F'),$$

$$(7) \quad \widetilde{\lambda}'(\sigma)(F)(v) = F(\sigma^{-1}(v)).$$

Recall, for $g \in \mathbf{Sp}_V(F')\mathbf{H}_V(F')$, we have

$$\widetilde{\rho}'(g)\widetilde{\rho}'(\sigma)(f)(x^\star) = \rho'(g)I_\sigma(f)(x^\star),$$

where $x^\star \in \mathbf{X}^\star(F')$. As shown in [Ge], $\widetilde{\rho}'|_{\mathbf{Sp}_V(F')\mathbf{H}_V(F')}$ is isomorphic to the Weil representation ρ'^- associated to the character ψ^- (defined as $x \rightarrow \psi(-x)$) of F . Hence the extended representation $\widetilde{\rho}'$ can be realized in $\mathbb{C}[\mathbf{X}(F')]$ by the analogous formula:

$$\widetilde{\rho}'(g)\widetilde{\rho}'(\sigma)(f)(x) = e^{\frac{2\pi i k'}{m}} \rho'^-(g)I_\sigma(f)(x)$$

for $g \in \mathbf{Sp}_V(F')\mathbf{H}_V(F')$, $x \in \mathbf{X}(F')$ and m -th root of unity ξ . Computing its trace at σ , we get $\xi = 1$.

Now let I be an automorphism on $\mathbb{C}[\mathbf{V}(F')]$ defined by $I(f)(x + x^\star) = \psi'(\langle x, x^\star \rangle)f(x + x^\star)$ for $x \in \mathbf{X}(F')$ and $x^\star \in \mathbf{X}^\star(F')$. In [[Ge], p. 84], Gérardin verifies that $I \cdot \widetilde{\lambda}'(h) = \check{\rho}' \otimes \rho'(h) \cdot I$ for $h \in \mathbf{H}_V(F')$. Moreover, Gérardin observes that any other such endomorphism I' is the composition of a convolution operator $\phi \star$ on $\mathbb{C}[\mathbf{V}(F')]$ with I . If one takes this ϕ :

$$\phi(x + x^\star) := \psi'(2\langle x^\star, x \rangle) \text{ for } x \in \mathbf{X}(F'), x^\star \in \mathbf{X}^\star(F'),$$

then the results in [[Ge], p.85] say that

$$I' \widetilde{\lambda}'(s) = \rho'^- \otimes \rho'(s)I' \text{ for } s \in \mathbf{Sp}_V(F').$$

Moreover, by definition, we see $I' \circ I_\sigma = I_\sigma \circ I'$, so $I' \cdot \widetilde{\lambda}'(\sigma)f = I'I_\sigma(f) = I_\sigma I'(f) = \rho'^- \otimes \rho'(\sigma)I'(f)$ for $f \in \mathbf{V}(F')$, and the result follows. \square

Corollary 5.2. $\text{tr } \widetilde{\rho}'$ is 0 outside the conjugates of $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{Z}_V(F')$.

6. ORTHOGONAL DECOMPOSITION

Let $V = V_1 \oplus V_2$ be a decomposition of V into the direct orthogonal sum of two symplectic spaces V_1 and V_2 . We then have a group homomorphism

$$\begin{aligned} \mathbf{H}_{V_1}(F) \times \mathbf{H}_{V_2}(F) &\longrightarrow \mathbf{H}_V(F) \\ [(v_1, k_1), (v_2, k_2)] &\longmapsto (v_1 + v_2, k_1 + k_2), \end{aligned}$$

and an obvious embedding

$$\mathbf{Sp}_{V_1}(F) \times \mathbf{Sp}_{V_2}(F) \longrightarrow \mathbf{Sp}_V(F),$$

so we get a group homomorphism

$$(\mathbf{Sp}_{V_1}(F)\mathbf{H}_{V_1}(F)) \times (\mathbf{Sp}_{V_2}(F)\mathbf{H}_{V_2}(F)) \xrightarrow{\delta} \mathbf{Sp}_V(F)\mathbf{H}_V(F).$$

It is a result of [Ge] that $\rho \circ \delta$ is isomorphic to the (external) tensor product of the Weil representations ρ_1, ρ_2 associated to ψ and the symplectic spaces V_1, V_2 .

Over F' , we have analogously a group homomorphism

$$\mathbf{Sp}_{V_1}(F')\mathbf{H}_{V_1}(F') \times \mathbf{Sp}_{V_2}(F')\mathbf{H}_{V_2}(F') \longrightarrow \mathbf{Sp}_V(F')\mathbf{H}_V(F').$$

It clearly extends to a group homomorphism

$$\delta' : \Gamma \ltimes [\mathbf{Sp}_{V_1}(F')\mathbf{H}_{V_1}(F') \times \mathbf{Sp}_{V_2}(F')\mathbf{H}_{V_2}(F')] \longrightarrow \Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{H}_V(F'),$$

and the left hand side is a subgroup of $(\Gamma \ltimes \mathbf{Sp}_{V_1}(F')\mathbf{H}_{V_1}(F')) \times (\Gamma \ltimes \mathbf{Sp}_{V_2}(F')\mathbf{H}_{V_2}(F'))$. We write $\widetilde{\rho}'_1, \widetilde{\rho}'_2$ for the extended Weil representations of the two components of that groups.

Proposition 6.1. *The representation $\widetilde{\rho}' \circ \delta'$ is isomorphic to the restriction of $\widetilde{\rho}'_1 \otimes \widetilde{\rho}'_2$ to $\Gamma \ltimes [(\mathbf{Sp}_{V_1}(F')\mathbf{H}_{V_1}(F')) \times (\mathbf{Sp}_{V_2}(F')\mathbf{H}_{V_2}(F'))]$.*

Proof. On restriction to $(\mathbf{Sp}_{V_1}(F')\mathbf{H}_{V_1}(F')) \times (\mathbf{Sp}_{V_2}(F')\mathbf{H}_{V_2}(F'))$, that is the above mentioned result of Gérardin. To compare the two extensions to the semi-direct product with Γ , it is enough to compare the traces at σ (provided they are non-zero). If $\dim V_1 = 2n_1$, and $\dim V_2 = 2n_2$, the trace of $\widetilde{\rho}' \circ \delta'$ at σ is q^n , and the trace of $\widetilde{\rho}'_1 \otimes \widetilde{\rho}'_2$ at (σ, σ) is $q^{n_1} q^{n_2} = q^n$. \square

7. RESTRICTION TO A PROPER PARABOLIC SUBGROUP

Let now V_+ be a non-trivial isotropic subspace of V , and V_0 the symplectic space V_+^\perp/V_+ . Write P for parabolic subgroup of Sp_V which is the stabilizer of V_+ . Write the corresponding linear algebraic groups over F as \mathbf{V}_+ , \mathbf{V} , \mathbf{V}_0 , \mathbf{P} , \mathbf{Sp}_V . Then we have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{U} \longrightarrow \mathbf{P} \longrightarrow \mathbf{GL}_{V_+} \times \mathbf{Sp}_{V_0} \longrightarrow 1,$$

where \mathbf{U} is the unipotent radical of \mathbf{P} and the homomorphisms $\mathbf{P} \longrightarrow \mathbf{GL}_{V_+}$, $\mathbf{P} \longrightarrow \mathbf{Sp}_{V_0}$ are given by the induced actions on \mathbf{V}_+ , \mathbf{V}_+^\perp . Note that when V_+ is a maximal isotropic subspace, $\mathbf{V}_0 = \{0\}$ and \mathbf{Sp}_{V_0} is just the trivial group.

Let H_\perp be the inverse image of V_+^\perp in H_V , then the Heisenberg group H_{V_0} appears as the quotient of H_\perp by the subgroup $V_+ \oplus 0$. Of course, P stabilizes H_\perp . All are viewed as algebraic groups over F , and denoted by the bold letters. It follows that we have a natural homomorphism

$$\mathbf{P} \mathbf{H}_\perp \xrightarrow{\eta} \mathbf{GL}_{V_+} \times (\mathbf{Sp}_{V_0} \mathbf{H}_{V_0}).$$

Writing ρ_0 for the Weil representation of $\mathbf{Sp}_{V_0}(F) \mathbf{H}_{V_0}(F)$ associated to ψ (the trivial representation of the trivial group if $\mathbf{V}_0(F) = 0$), and ϵ for the unique non-trivial quadratic character F^\times , it is a result of Gérardin[[Ge], Theorem 2.4] that the restriction of ρ to $\mathbf{P}(F) \mathbf{H}_V(F)$ is induced from the representation of $\mathbf{P}(F) \mathbf{H}_\perp(F)$ obtained by (composing with η over F) the representation given by $\epsilon \circ \det$ on $\mathbf{GL}_{V_+}(F)$ and by ρ_0 on $\mathbf{Sp}_{V_0}(F) \mathbf{H}_{V_0}(F)$.

Now η over F' readily extends to a homomorphism

$$\eta' : \Gamma \ltimes (\mathbf{P}(F') \mathbf{H}_\perp(F')) \longrightarrow \Gamma \ltimes (\mathbf{GL}_{V_+}(F') \mathbf{Sp}_{V_0}(F') \mathbf{H}_{V_0}(F')),$$

and the group on the right is a subgroup of $(\Gamma \ltimes \mathbf{GL}_{V_+}(F')) \times (\Gamma \ltimes \mathbf{Sp}_{V_0}(F') \mathbf{H}_{V_0}(F'))$.

Now $\tilde{\epsilon}' \circ \det : \Gamma \ltimes \mathbf{GL}_{V_+}(F') \longrightarrow \mathbb{C}^\times$; $(\tau, g) \mapsto \epsilon'(\det(g))$ is a character of $\Gamma \ltimes \mathbf{GL}_{V_+}(F')$. We can take the product of $\tilde{\epsilon}' \circ \det$ with the extended Weil representation $\tilde{\rho}'_0$ of $\Gamma \ltimes \mathbf{Sp}_{V_0}(F') \mathbf{H}_{V_0}(F')$ associated to ψ' (or the trivial representation of Γ if $\mathbf{V}_0 = \{0\}$), and restrict to $\Gamma \ltimes (\mathbf{GL}_{V_+}(F') \times \mathbf{Sp}_{V_0}(F') \mathbf{H}_{V_0}(F'))$ to get a representation, written $\epsilon' \tilde{\rho}'_0$.

Proposition 7.1. *The restriction of $\tilde{\rho}'$ to $\Gamma \ltimes \mathbf{P}(F') \mathbf{H}_V(F')$ is induced from the representation $\epsilon' \tilde{\rho}'_0$ composed with η' .*

Proof. On restriction to $\mathbf{P}(F') \mathbf{H}_V(F')$, that is the above mentioned result of Gérardin. As in §6, it is enough to compute the trace at σ . So we need to check that the trace of the induced representation at σ is indeed q^n . Now the cosets in $\Gamma \ltimes \mathbf{P}(F') \mathbf{H}_V(F') / \Gamma \ltimes \mathbf{P}(F') \mathbf{H}_\perp(F')$ are represented by $\mathbf{H}_V(F') / \mathbf{H}_\perp(F')$ and for $h \in \mathbf{H}_V(F')$, we have

$$(1, h)(\sigma, 1)(1, h^{-1}) = (\sigma, h\sigma(h^{-1})) = (\sigma, 1)(1, \sigma^{-1}(h)h^{-1}),$$

which belongs to $\Gamma \ltimes \mathbf{P}(F') \mathbf{H}_\perp(F')$ only if h is fixed by σ modulo $\mathbf{H}_\perp(F')$; but then this means that we can take h to be in $\mathbf{H}_V(F)$, in which case, $(1, h)(\sigma, 1)(1, h^{-1}) = (\sigma, 1)$. All in all, the trace of the induced representation at σ is $|\mathbf{H}_V(F) / \mathbf{H}_\perp(F)| \cdot q^{n_0}$ with $\dim V_0 = 2n_0$; since $|\mathbf{H}_V(F) / \mathbf{H}_\perp(F)| = |\mathbf{V}_+(F)| = q^{n-n_0}$, we get the desired result. \square

8. REDUCTIONS

We now start the proof of equality (\star) in the main theorem. We proceed by induction on $n = \frac{1}{2} \dim_F V$. The case where $n = 0$ being entirely trivial, we assume $n > 0$. We fix i and t , and as remarked in §3, we may and do assume that i is prime to m , so $d = 1$. We have to prove (\star) for a fixed g , or equivalently for a fixed norm $h = N_{i,t}(\sigma^i, g)$.

If h does not belong to $\mathbf{Sp}_V(F) \mathbf{Z}_V(F)$, then (σ^i, g) is not conjugate to $\sigma^i \ltimes \mathbf{Sp}_V(F') \mathbf{Z}_V(F')$. In that case, the equality (\star) is $0 = 0$ by the result of §5. So we may assume that h belongs to $\mathbf{Sp}_V(F) \mathbf{Z}_V(F)$. But $\mathbf{Z}_V(F')$ acts in $\tilde{\rho}'$ via the character ψ' , so on $\Gamma \ltimes \mathbf{Z}_V(F')$, the character relation (\star) is immediate. Applying Lemma 3.1 (iii) to the product $\mathbf{Sp}_V(F) \mathbf{Z}_V(F)$, we see that we may assume that h belongs to $\mathbf{Sp}_V(F)$, and g to $\mathbf{Sp}_V(F')$.

If h stabilizes a non-trivial decomposition $V = V_1 \oplus V_2$ as in §6, it acts on $\mathbf{V}(F)$ via (h_1, h_2) with $h_1 \in \mathbf{Sp}_{V_1}(F)$, $h_2 \in \mathbf{Sp}_{V_2}(F)$. By Lemma 3.1 (iii) and Proposition 6.1, the equality (\star) comes from the induction hypothesis

applied to V_1 and h_1 in $\mathbf{Sp}_{V_1}(F)$, and to V_2 and h_1 in $\mathbf{Sp}_{V_2}(F)$.

If h stabilizes a non-trivial totally isotropic subspace V_+ of V , then it belongs to the group $\mathbf{P}(F)$ of §7, and we can take g in $\mathbf{P}(F')$, write (g_+, g_0) for the projection of g to $\mathbf{GL}_{V_+}(F') \times \mathbf{Sp}_{V_0}(F')$, and similarly (h_+, h_0) for h . We note that

$$\mathrm{tr} \widetilde{\rho}'_0(\sigma^i, g_0) = \mathrm{tr} \rho_0(h_0)$$

by the induction hypothesis and

$$\widetilde{e}'(\det g_+) = e(\det h_+)$$

directly. The equality (\star) for (g, h) then comes from Proposition 7.1 and Lemma 3.1 applied to the induction from \mathbf{PH}_\perp to \mathbf{PH}_V .

So the only remaining case is when h stabilizes no non-trivial orthogonal decomposition $V = V_1 \oplus V_2$, and stabilizes no non-trivial isotropic subspace V_+ of V . Let us analyze that case. Let $h = su$ be the Jordan decomposition of h into a semi-simple part s and a unipotent part u , with $su = us$. Then $F[s]$ is a semi-simple commutative subalgebra of $\mathrm{End}_F(V)$, and the adjoint involution on $\mathrm{End}_F(V)$ associated to the symplectic form on V induces $s \mapsto s^{-1}$ on $F[s]$.

Writing $F[s]$ as a product of fields $(F_\alpha)_{\alpha \in A}$, we accordingly have a decomposition of V as a direct sum $V = \bigoplus_{\alpha \in A} V_\alpha$, where $F[s]$ acts on V_α via F_α . The involution $s \mapsto s^{-1}$ gives a permutation $\alpha \mapsto \bar{\alpha}$ on A , together with isomorphisms $F_\alpha \simeq F_{\bar{\alpha}}$, and the orthogonal V_α^\perp of V_α is $\bigoplus_{\beta \neq \alpha} V_\beta$.

Assume first that A has at least two elements, and take α in A . If $\alpha = \bar{\alpha}$ then V is the orthogonal direct sum of V_α and $\bigoplus_{\beta \neq \alpha} V_\beta$; each of those subspaces is stable under s and u , hence under h , which contradicts our assumption on h . If $\alpha \neq \bar{\alpha}$, then h stabilizes the non-trivial totally isotropic subspace V_α , which again contradicts our assumption on h .

So we see that A has only one element, say $\alpha = \bar{\alpha}$. So $F[s]$ is a field E , and u is an E -linear endomorphism of V , the involution $s \mapsto s^{-1}$ on E has a fixed subfield E_+ .

Assume first that $E = E_+$, i.e. $s = \pm 1$, which implies $E = F$; then $\mathrm{Ker}(u - 1_V)$ is a non-trivial subspace of V , and any line in that subspace is isotropic and stable under h , again a contradiction. We conclude that E is a quadratic extension of E_+ ; then there exists a skew-hermitian form φ on the E -vector space V —skew-hermitian with respect to E/E_+ such that, for v, v' in V ,

$$\langle v, v' \rangle = \mathrm{tr}_{E_+/F}(\varphi(v, v')).$$

Then s acts on V as an element of E^\times with norm 1 to E_+ , and u acts as a unipotent element of the unitary group associated to φ . Now the kernel of $u - 1_V$ is orthogonal to its image. If $u \neq 1_V$, then the intersection $\mathrm{Im}(u - 1_V) \cap \mathrm{Ker}(u - 1_V)$ is a non-zero isotropic subspace of V stable under $h = su$.

So we conclude that $u = 1_V$ and that the E -vector space V contains no isotropic non-zero vector with respect to φ : that implies V has dimension 1 over E . This very special case will be treated in the next §9 and §10.

9. THE CASE OF SL_2

We keep the preceding notation, and write \mathbf{U}_φ for the unitary group of φ seen as an algebraic group over E_+ , and \mathbf{T} for its restriction of scalars from E_+ to F . Thus \mathbf{T} is a maximally elliptic torus of \mathbf{Sp}_V over F , and $\mathbf{T}(F) = \mathbf{U}_\varphi(E_+)$ is the group E^1 of elements of E with norm 1 to E_+ .

Let ω be the non-trivial character of $\mathbf{T}(F)$ of order 2, and $\omega\psi$ the character of $\mathbf{T}(F)\mathbf{Z}_V(F)$ given by ω on $\mathbf{T}(F)$ and ψ on $\mathbf{Z}_V(F)$.

Proposition 9.1. *The virtual representation $\nu = \mathrm{Ind}_{\mathbf{H}_V(F)}^{\mathbf{T}(F)\mathbf{H}_V(F)}(\rho|_{\mathbf{H}_V(F)}) - \mathrm{Ind}_{\mathbf{T}(F)\mathbf{Z}_V(F)}^{\mathbf{T}(F)\mathbf{H}_V(F)}\omega\psi$ is the restriction of ρ to $\mathbf{T}(F)\mathbf{H}_V(F)$.*

Proof. The first term of the virtual representation ν is the sum of the inequivalent irreducible representations $\varphi\rho|_{\mathbf{T}(F)\mathbf{H}_V(F)}$ where φ runs through all characters of $\mathbf{T}(F)$. For such a character φ , the multiplicity of $\varphi\rho|_{\mathbf{T}(F)\mathbf{H}_V(F)}$

in the second term of the virtual representation is the multiplicity of ω in $\varphi\rho|_{\mathbf{T}(F)}$. But it follows from [[Ge], p.73] that $\rho|_{\mathbf{T}(F)}$ is the direct sum of the characters of $\mathbf{T}(F)$ distinct from ω , hence the result. \square

The situation we are reduced to is the following: we have an element s of $\mathbf{T}(F)$ which is the norm from F' to F of some element s' of $\mathbf{T}(F')$ (note that \mathbf{T} is commutative), and we want to show that

$$(\star\star) \quad \mathrm{tr} \widetilde{\rho}'(\sigma^i, s') = \mathrm{tr} \rho(s),$$

for any integer i , $0 \leq i \leq m-1$, prime to m . Note also that $E = F[s]$ so that in particular s and s' are not 1.

It is tempting to try and prove it via a proposition similar to the above, but for $\widetilde{\rho}'|_{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')}$. That is not so straightforward, essentially because the F' -algebra $E \otimes_F F'$ is generally no longer a field. In this section, we will treat the case $n = 1$ so that $E_+ = F$; the general case will be dealt with in §10.

First we assume that m is odd; then $E \otimes_F F'$ is a field E' —a quadratic extension of F' . We denote by ω' the order 2 character of $\mathbf{T}(F')$, which is simply ω composed with the norm from $\mathbf{T}(F')$ to $\mathbf{T}(F)$, since that norm is surjective.

Proposition 9.2. *Assume $n = 1$ and m odd. Then the virtual representation*

$$\nu' = \mathrm{Ind}_{\Gamma \ltimes \mathbf{H}_V(F')}^{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')} (\widetilde{\rho}'|_{\Gamma \ltimes \mathbf{H}_V(F')}) - \mathrm{Ind}_{\Gamma \ltimes \mathbf{T}(F')\mathbf{Z}_V(F')}^{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')} (\omega' \widetilde{\psi}')$$

is the restriction of $\widetilde{\rho}'$ to $\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')$. Here $\omega' \widetilde{\psi}'$ is the character of $\Gamma \ltimes \mathbf{T}(F')\mathbf{Z}_V(F')$ obtained by extending $\omega' \psi'$ trivially on Γ .

Let us assume Proposition 9.2 for a moment, and prove $(\star\star)$ in our special case $n = 1$, m odd. As s and s' are not 1, the first term of the virtual representations contribute nothing to $\mathrm{tr} \widetilde{\rho}'(\sigma^i, s')$ and $\mathrm{tr} \rho(s)$. But it is clear that $\omega \psi$ and $\omega' \psi'$ verify the Shintani relation for the lifting from $\mathbf{T}(F)\mathbf{Z}_V(F)$ to $\Gamma \ltimes \mathbf{T}(F')\mathbf{Z}_V(F')$. By Lemma 3.1, it follows that the second terms have equal contribution, which gives $(\star\star)$.

Let us now prove Proposition 9.2: we remark that ν' has positive dimension and that $\mathrm{tr} \nu'(\sigma) = q^m = \mathrm{tr} \widetilde{\rho}'(\sigma)$. The following lemma then shows that ν' is an irreducible representation. By Proposition 9.1, with F' as a base field, we see that ν' is an extension of $\widetilde{\rho}'|_{\mathbf{T}(F')\mathbf{H}_V(F')}$, and Proposition 9.2 follows from the equality of traces at σ .

Lemma 9.3. $\langle \nu', \nu' \rangle = 1$.

Proof. By Lemma 3.1, $\langle \nu', \nu' \rangle = \frac{1}{|\mathrm{Gal}(F'/F)\mathbf{T}(F')\mathbf{H}(F')|} \sum_{i=0}^{m-1} \sum_{A \in \mathbf{T}(F')\mathbf{H}(F')} \nu'(\sigma^i, A) \overline{\nu'(\sigma^i, A)}$
 $= \frac{1}{|\mathrm{Gal}(F'/F)\mathbf{T}(F')\mathbf{H}(F')|} \sum_{i=0}^{m-1} |\sigma^i \ltimes \mathbf{T}(F')\mathbf{H}(F')| \langle \mathcal{N}_{i,t}(\nu'), \mathcal{N}_{i,t}(\nu') \rangle$
 $= \frac{1}{|\mathrm{Gal}(F'/F)\mathbf{T}(F')\mathbf{H}(F')|} \sum_{i=0}^{m-1} |\sigma^i \ltimes \mathbf{T}(F')\mathbf{H}(F')| \langle \mathcal{N}_{i,t}(\nu'), \mathcal{N}_{i,t}(\nu') \rangle$
 $= \frac{1}{|\mathrm{Gal}(F'/F)\mathbf{T}(F')\mathbf{H}(F')|} \sum_{i=0}^{m-1} |\sigma^i \ltimes \mathbf{T}(F')\mathbf{H}(F')| \langle \rho_{(m,i)}, \rho_{(m,i)} \rangle = 1. \quad \square$

Now, we assume that m is even, still with $n = 1$; then $E \otimes_F F'$ splits as $F' \oplus F'$, so that $\mathbf{T}(F')$ is isomorphic to F'^{\times} . As before, we let ω' be the order 2 character of $\mathbf{T}(F')$, which is again ω composed with the norm map from $\mathbf{T}(F')$ to $\mathbf{T}(F)$. Now we write η for the order 2 character of Γ , and extend as before $\omega' \psi'$ to a character of $\Gamma \ltimes \mathbf{T}(F')\mathbf{Z}_V(F')$ trivial on Γ .

Proposition 9.4. *Assume $n = 1$ and m even. Then the virtual representation*

$$\nu' = -\mathrm{Ind}_{\Gamma \ltimes \mathbf{H}_V(F')}^{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')} (\widetilde{\rho}'|_{\Gamma \ltimes \mathbf{H}_V(F')}) + \mathrm{Ind}_{\Gamma \ltimes \mathbf{T}(F')\mathbf{Z}_V(F')}^{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')} \omega' \widetilde{\psi}'$$

is the restriction of $\eta \widetilde{\rho}'$ to $\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')$.

Proof. By construction, $\dim \nu' = q^m$ and from Lemma 3.1, we get $\mathrm{tr} \nu'(\sigma) = -q$. The following lemma, proved as above, then shows that ν' is irreducible. By Proposition 9.1, it is an extension of $\widetilde{\rho}'|_{\mathbf{T}(F')\mathbf{H}_V(F')}$, which has to be $\eta \widetilde{\rho}'$ since $\mathrm{tr} \nu'(\sigma) = -q$. \square

Lemma 9.5. $\langle \nu', \nu' \rangle = 1$.

We can now prove $(\star\star)$ when $n = 1$ and m is even. The proof is as above taking signs in account; the first terms in $\mathrm{tr} \eta \nu'(\sigma^i, s')$ and $\mathrm{tr} \nu(s)$ contribute nothing, and the second terms are equal, because $\eta(\sigma^i) = -1$.

10. END OF PROOF

We are now ready to prove the formula (★★) for a general $n \geq 1$ (we keep the notation and assumptions of §8). We proceed by a kind of reduction to §9, and the problem is rather a matter of careful book-keeping.

We can see V as a vector space over E_+ . Endowing it with the form $\delta = \text{tr}_{E/E_+} \varphi$, we get a symplectic vector space of dimension 2 over E_+ , for which we write W ; with it comes a Heisenberg group \mathbf{H}_W with $\mathbf{H}_W(E_+) = \mathbf{W}(E_+) \oplus E_+$ as sets, and there is an obvious morphism from $\mathbf{H}_W(E_+)$ to $\mathbf{H}_V(F)$ which is identity on $\mathbf{W}(E_+) = \mathbf{V}(F)$ and is given by $\text{tr}_{E_+/F}$ on $\mathbf{Z}_W(E_+)$. On the other hand $\mathbf{Sp}_W(E_+)$ is obviously a subgroup of $\mathbf{Sp}_V(F)$. That gives a morphism r from $\mathbf{Sp}_W(E_+)\mathbf{H}_W(E_+)$ to $\mathbf{Sp}_V(F)\mathbf{H}_V(F)$. If we write ρ_W for the Weil representation of $\mathbf{Sp}_W(E_+)\mathbf{H}_W(E_+)$ associated with $\psi_{E_+} = \psi \circ \text{tr}_{E_+/F}$, we know [[Ge], 2.6] that $\rho \circ r$ is isomorphic to ρ_W . As $\mathbf{T}(F)$ is included in $\mathbf{Sp}_W(E_+)$, we can work with $\rho_W|_{\mathbf{T}(F)\mathbf{H}_W(E_+)}$ rather than $\rho|_{\mathbf{T}(F)\mathbf{H}_V(F)}$. Similarly, we want to express $\widetilde{\rho'}|_{\Gamma \ltimes \mathbf{T}(F')\mathbf{H}_V(F')}$ in terms of Weil representations attached to 2-dimensional symplectic spaces.

Let e be the greatest common divisor of m and n ; then $E_+ \otimes_F F'$ splits as the direct sum of e fields E_+^α , each of degree m/e over E_+ and n/e over F' . The group Γ permutes the factors transitively, and the stabilizer of each factor is generated by σ^e ; more precisely, σ induces an E_+ -linear isomorphism of E_+^α to $(E_+^\alpha)^\sigma$, and σ^e gives a generator of $\text{Gal}(E_+^\alpha/E_+)$ for each α . Note however that σ^e is not in general the Frobenius automorphism of E_+^α/E_+ ; that will not cause any problem with norms, nevertheless, because \mathbf{T} is commutative.

Now $\mathbf{W}(E_+ \otimes_F F')$ is endowed with a $E_+ \otimes_F F'$ -bilinear symplectic form (obtained from δ by scalar extension); it splits as a direct sum of spaces W_α ; each has dimension 2 over E_+^α and carries the E_+^α -bilinear symplectic form δ_α obtained from δ by scalar extension from E_+ to E_+^α . The symplectic space $\mathbf{W}(E_+ \otimes_F F')$ is the orthogonal direct sum of the symplectic subspaces W_α . Endowed with the F' -bilinear symplectic form $\text{tr}_{E_+/F'}(\delta_\alpha)$, W_α is a symplectic vector space V_α over F' , and $\mathbf{V}(F')$ is isomorphic to the orthogonal direct sum of the V_α 's.

Now for each α , $\mathbf{Sp}_{V_\alpha}(F')$ is a subgroup of $\mathbf{Sp}_V(F')$ and we have a natural inclusion $\mathbf{H}_{V_\alpha}(F') \rightarrow \mathbf{H}_V(F')$. Altogether, that gives a morphism $\prod_\alpha \mathbf{Sp}_{V_\alpha}(F')\mathbf{H}_{V_\alpha}(F') \rightarrow \mathbf{Sp}_V(F')\mathbf{H}_V(F')$ and it follows from [[Ge], 4.6] that the inflation of ρ' through that morphism is the tensor product of the Weil representations ρ'_α (with respect to ψ').

Similarly, through the natural morphism from $\mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha)$ to $\mathbf{Sp}_{V_\alpha}(F')\mathbf{H}_{V_\alpha}(F')$, ρ'_α gives the Weil representation ρ_α^+ of $\mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha)$ attached to $\psi \circ \text{tr}_{E_+/F'} = \psi_{E_+} \circ \text{tr}_{E_+/E_+}$.

We now want to extend $\otimes_\alpha \rho_\alpha^+$ to a representation R of $\Gamma \ltimes (\prod_\alpha \mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha))$ giving trace q^n to σ . We use tensor induction for that. More precisely, enumerate the α 's as $E_+^{\alpha_0}, E_+^{\alpha_1} = [E_+^{\alpha_0}]^\sigma, \dots, E_+^{\alpha_{e-1}} = [E_+^{\alpha_0}]^{\sigma^{e-1}}$ (with $E_+^{\alpha_e} = [E_+^{\alpha_0}]^{\sigma^e}$) and fix a model for $\rho_{\alpha_0}^+$ on some space X , for example a Schrödinger model; extend that model uniquely to a representation $\widetilde{\rho}_\alpha^+$ of $\langle \sigma^e \rangle \ltimes \mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha)$, so that σ^e has trace q^n (here $\langle \sigma^e \rangle$ denotes the subgroup of Γ generated by σ^e). Then there is a unique action of $\Gamma \ltimes (\prod_\alpha \mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha))$ on $X_0 \otimes X_1 \otimes \dots \otimes X_{e-1}$, with $X_i = X$ for $i = 0, \dots, e-1$, such that

- (1) For $i = 0, \dots, e-1$, $\mathbf{Sp}_{W_{\alpha_i}}(E_+^{\alpha_i})\mathbf{H}_{W_{\alpha_i}}(E_+^{\alpha_i})$ acts only on the factor X_i via $\rho_{\alpha_0}^+ \circ \sigma^{-i}$.
- (2) σ acts by sending $x_0 \otimes \dots \otimes x_{e-1}$ to $(\sigma^e(x_{e-1}) \otimes x_0 \otimes \dots \otimes x_{e-2})$.

Clearly the trace of σ on that representation is the trace of σ^e on X i.e. q^n , and the restriction to $\prod_\alpha \mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha)$ is isomorphic to the product of ρ_α^+ . It follows that if we inflate $\widetilde{\rho'}$ via the natural homomorphism from $\Gamma \ltimes (\prod_\alpha \mathbf{Sp}_{W_\alpha}(E_+^\alpha)\mathbf{H}_{W_\alpha}(E_+^\alpha))$ to $\Gamma \ltimes \mathbf{Sp}_V(F')\mathbf{H}_V(F')$, then get a representation isomorphic to R .

Now let us return to our element s in $\mathbf{T}(F) = E^1 \subseteq E^\times$; we rather see \mathbf{T} as a maximally elliptic torus \mathbf{S} of \mathbf{Sp}_W , so that s is an element of $\mathbf{Sp}_W(E_+)$; the symplectic vector space W_{α_0} is obtained from W by scalar extension from E_+ to $E_+^{\alpha_0}$, and there is an element s'_0 of $\mathbf{S}(E_+^{\alpha_0})$ with norm s to $\mathbf{S}(E_+) = E^1$. Now consider the element $s' = (s'_0, 1, \dots, 1)$ of $\mathbf{S}(E_+ \otimes F') = \prod_{j=0}^{e-1} \mathbf{S}(E_+^{\alpha_j})$; then we have $N_{i,t}(\sigma^i, s') = s$ (norm from $\mathbf{T}(F')$ to $\mathbf{T}(F)$). But from

§9, treating the case where n is 1, we see that

$$\mathrm{tr} \widetilde{\rho_{\alpha_0}^+}(\sigma^{ie}, s'_0) = \mathrm{tr} \widetilde{\rho_W}(s).$$

From the construction of R , we see that

$$\mathrm{tr} R(\sigma^i, s') = \mathrm{tr} \widetilde{\rho_{\alpha_0}^+}(\sigma^{ie}, s'_0),$$

and the result follows from §9.

Remark 10.1. *Clearly, the considerations of this section have to do with the behaviour of Gyoja's norm maps with respect to restriction of scalars. As our concern is more immediate, we have refrained from developing that aspect along the lines of [D2].*

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